Imperial College London

Lecture 16

More z-Transform

(Lathi 5.2,5.4-5.5)

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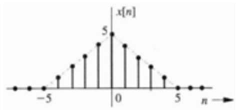
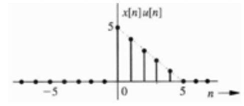
Shift Property of z-Transform

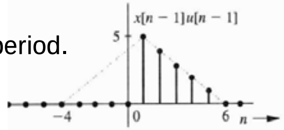
- If $x[n]u[n] \Longleftrightarrow X[z]$

  then $x[n-1]u[n-1] \Longleftrightarrow \frac{1}{z}X[z]$



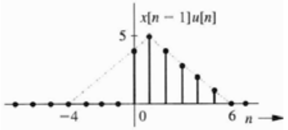


  which is delay causal signal by 1 sample period.



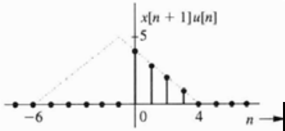
- If we delay x[n] first:

  $$x[n-1]u[n] \Longleftrightarrow \frac{1}{z}X[z] + x[-1]$$



- If we ADVANCE x[n] by 1 sample period:

  $$x[n+1]u[n] \Longleftrightarrow zX[z] - zx[0]$$



L5.2 p508

Convolution property of z-transform

- If h[n] is the impulse response of a discrete-time LTI system, then

  then $H[z] = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ $\qquad h[n] \Longleftrightarrow H[z]$

- If $x_1[n] \Longleftrightarrow X_1[z]$ and $x_2[n] \Longleftrightarrow X_2[z]$,
- Then

  $$x_1[n] * x_2[n] \Longleftrightarrow X_1[z]X_2[z]$$

- That is: convolution in the time-domain is the same as multiplication in the z-domain.
- Therefore, we can derive the input-output relationship fo any LTI systems in z-domain:

  $$y[n] = x[n] * h[n]$$

  $$Y[z] = X[z]H[z]$$

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More Properties of z-Transform

- For all these cases, we assume: $x[n]u[n] \Longleftrightarrow X[z]$
- Scaling Property:

  $$\gamma^n x[n]u[n] \Longleftrightarrow X\left[\frac{z}{\gamma}\right]$$

- Multiply by n property:

  $$nx[n]u[n] \Longleftrightarrow -z\frac{d}{dz}X[z]$$

- Time reversal property:

  $$x[-n] \Longleftrightarrow X[1/z]$$

- Initial value property: $X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots$

  $$x[0] = \lim_{z\to\infty} X[z]$$

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Summary of z-transform properties (1)

Operation	$x[n]$	$X[z]$
Addition	$x_1[n]+x_2[n]$	$X_1[z]+X_2[z]$
Scalar multiplication	$ax[n]$	$aX[z]$
Right-shifting	$x[n-m]u[n-m]$	$\frac{1}{z^m}X[z]$
	$x[n-m]u[n]$	$\frac{1}{z^m}X[z]+\frac{1}{z^m}\sum_{n=1}^{m}x[-n]z^n$
	$x[n-1]u[n]$	$\frac{1}{z}X[z]+x[-1]$
	$x[n-2]u[n]$	$\frac{1}{z^2}X[z]+\frac{1}{z}x[-1]+x[-2]$
	$x[n-3]u[n]$	$\frac{1}{z^3}X[z]+\frac{1}{z^2}x[-1]+\frac{1}{z}x[-2]+x[-3]$

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Summary of z-transform properties (2)

Operation	$x[n]$	$X[z]$
Left-shifting	$x[n+m]u[n]$	$z^mX[z]-z^m\sum_{n=0}^{m-1}x[n]z^{-n}$
	$x[n+1]u[n]$	$zX[z]-zx[0]$
	$x[n+2]u[n]$	$z^2X[z]-z^2x[0]-zx[1]$
	$x[n+3]u[n]$	$z^3X[z]-z^3x[0]-z^2x[1]-zx[2]$
Multiplication by γ^n	$\gamma^n x[n]u[n]$	$X\left[\frac{z}{\gamma}\right]$
Multiplication by n	$nx[n]u[n]$	$-z\frac{d}{dz}X[z]$
Time convolution	$x_1[n]*x_2[n]$	$X_1[z]X_2[z]$
Time reversal	$x[-n]$	$X[1/z]$
Initial value	$x[0]$	$\lim_{z\to\infty}X[z]$
Final value	$\lim_{N\to\infty}x[N]$	$\lim_{z\to 1}(z-1)X[z]$ poles of $(z-1)X[z]$ inside the unit circle

L5.2 p514

Discrete LTI System and Difference Equation

- Consider a discrete time system where the input-output relation is described by: $y[n]-5y[n-1]+6y[n-2]=x[n]+3x[n-1]+5x[n-2]$
- This is known as a difference equation, where current output is dependent on current input x[n], and two previous inputs and outputs x[n-1], x[n-2], y[n-1] and y[n-2].
- Take z-transform on both sides and assume zero-state condition:

$$Y(z)-5z^{-1}Y(z)+6z^{-2}Y(z)=X(z)+3z^{-1}X(z)+5z^{-2}X(z)$$

$$\Rightarrow (1-5z^{-1}+6z^{-2})Y(z)=(1+3z^{-1}+5z^{-2})X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)}=H(z)=\frac{1+3z^{-1}+5z^{-2}}{1-5z^{-1}+6z^{-2}}$$

- The transfer function of a general Nth order causal discrete LTI system is:

$$H[z]=\frac{b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}$$

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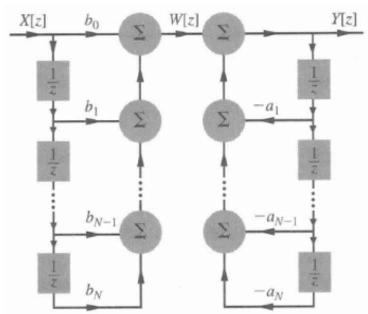
Realization of LTI System – Direct Form I

- The general transfer function H(z) can be realised using Direct Form I as follows:

$$Y[z]=H[z]X[z]=\frac{b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}X[z]$$

$$=\left(\frac{1}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}\right)\left(b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}\right)X[z]$$

$$=\left(\frac{1}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}\right)W(z)$$



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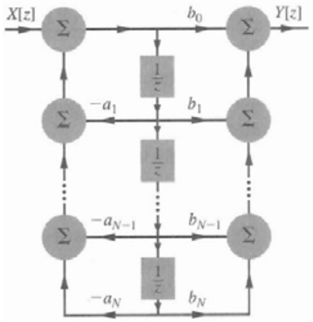
Realization of LTI System – Direct Form II

- Or use Canonical Director Form II:

$$Y[z]=H[z]X[z]=\frac{b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}X[z]$$

$$=\left(b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}\right)\left(\frac{1}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}\right)X[z]$$

$$=\left(b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}\right)W(z)$$



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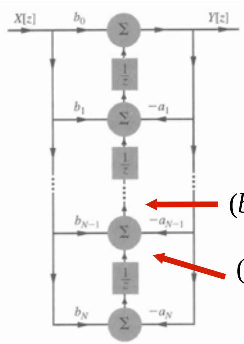
Realization of LTI System – Transposed Direct Form II

- Or the transposed version: $Y[z]=H[z]X[z]=\frac{b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}}{1+a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}}X[z]$

$$Y[z]+\left(a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}\right)Y[z]=\left(b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}\right)X[z]$$

$$Y[z]=\left(b_0+b_1z^{-1}+......+b_{N-1}z^{-N+1}+b_Nz^{-N}\right)X[z]-\left(a_1z^{-1}+......+a_{N-1}z^{-N+1}+a_Nz^{-N}\right)Y[z]$$



$(b_NX[z]-a_NY[z])z^{-1}+(b_{N-1}X[z]-a_{N-1}Y[z])$

$(b_NX[z]-a_NY[z])z^{-1}$

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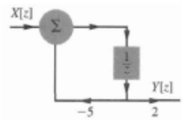
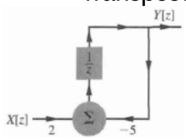
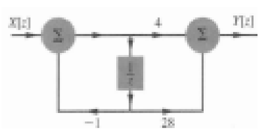
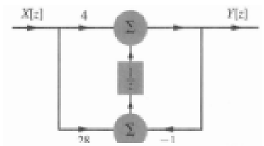
Examples

Direct Form II | Transposed

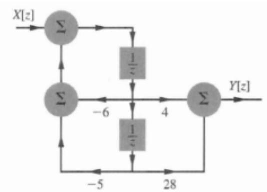
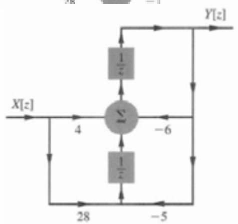
$$H[z]=\frac{2}{z+5}=\frac{2z^{-1}}{1+5z^{-1}}$$

$$H[z]=\frac{4z+28}{z+1}=\frac{4+28z^{-1}}{1+z^{-1}}$$

$$H[z]=\frac{4z+28}{z^2+6z+5}=\frac{4+28z^{-1}}{1+6z^{-1}+5z^{-2}}$$

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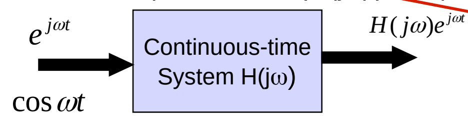
Frequency Response of Discrete-time Systems

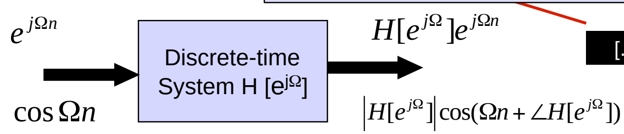
- Remember that for a continuous-time system, the system response to an input $e^{j\omega t}$ is $H(j\omega)\,e^{j\omega t}$.
- The response to an input $\cos \omega t$ is $|H(j\omega)| \cos(\omega t + \angle H(j\omega))$.



(.) continuous-time

- Now, consider a discrete-time system with z-domain transfer function H[z].
- Let $z = e^{j\Omega}$, the system response to an input $e^{j\Omega n}$ is $H[e^{j\Omega}]\,e^{j\Omega n}$.
- The response to an input $\cos \Omega n$ is $|H[e^{j\Omega}]| \cos(\cos \Omega n + \angle H[e^{j\Omega}])$.

$\Omega=\omega T_S$ where T_S is the sampling period.



[.] discrete-time

L5.5 p531

Frequency Response Example (1)

- For a system specified by the following difference equation, find the frequency response of the system.

$$y[n+1]-0.8y[n]=x[n+1]$$

- Take z-transform on both sides to find the transfer function:

$$zY[z]-0.8Y[z]=zX[z] \qquad \Rightarrow H[z]=\frac{Y[z]}{X[z]}=\frac{z}{z-0.8}=\frac{1}{1-0.8z^{-1}}$$

- Therefore the frequency response is:

$$H[e^{j\Omega}]=\frac{1}{1-0.8e^{-j\Omega}} = \frac{1}{(1-0.8\cos\Omega)+j0.8\sin\Omega}$$

$$|H[e^{j\Omega}]|=\frac{1}{\sqrt{(1-0.8\cos\Omega)^2+(0.8\sin\Omega)^2}} = \frac{1}{\sqrt{1.64-1.6\cos\Omega}}$$

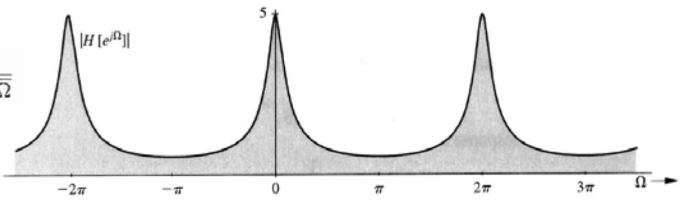
$$\angle H[e^{j\Omega}]=-\tan^{-1}\left[\frac{0.8\sin\Omega}{1-0.8\cos\Omega}\right]$$

L5.5 p533

Frequency Response Example (2)

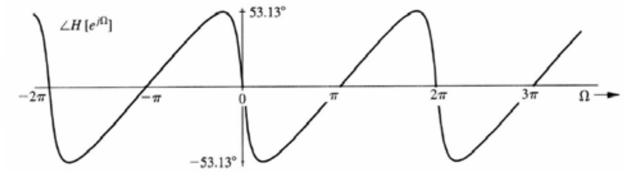
- Amplitude response

$$H[e^{j\Omega}]=\frac{1}{\sqrt{1.64-1.6\cos\Omega}}$$



- Phase response

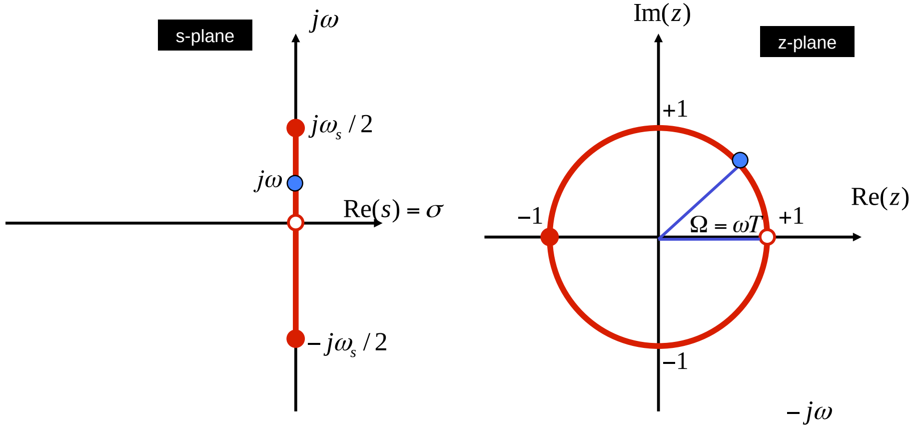
$$\angle H[e^{j\Omega}]=-\tan^{-1}\left[\frac{0.8\sin\Omega}{1-0.8\cos\Omega}\right]$$



L5.5 p533

Mapping from s-plane to z-plane

- Since $z=e^{sT}=e^{(\sigma+j\omega)T}=e^{\sigma T}e^{j\omega T}$ where T = $2\pi/\omega_s$ we can map the s-plane to the z-plane as below:



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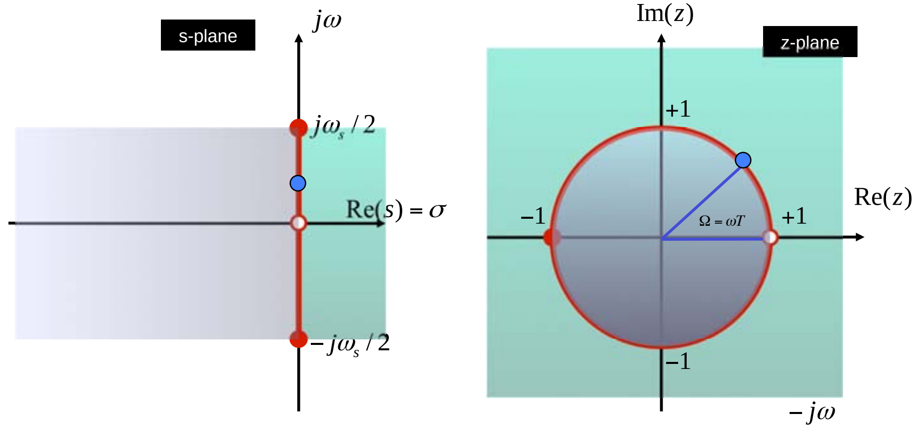

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